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CONTINUOUS MATRICES, ALGEBRAIC CORRESPONDENCES, AND CLOSURE.

By Albert A. Bennett.

I. Introduction.

This article is intended merely to show the intimate relation that exists between three topics not hitherto generally associated. It is first shown how an algebraic correspondence constitutes a type of continuous matrix, and how a continuous matrix serves in turn to suggest at once a weighting in correspondences, a notion that arises in a restricted manner from geometrical studies. The distinction between the inverse and the transposed correspondences, is a feature of the study of correspondences from an algebraic rather than a transcendental viewpoint, being determined not by the character at a point but by the nature of the correspondence as a whole. The condition that an algebraic correspondence has in the sense here used an algebraic inverse, leads at once to finite configurations and problems of closure.

It may be remarked in passing that of the obvious geometric problems of classification of types, one of the most promising, and one as yet wholly untouched, is that of finite variable algebraic configurations, or its equivalent problem of algebraic correspondences with algebraic inverses, the term inverse being understood as here defined. References are contained in the author's paper " Closed Algebraic Correspondences," *Annals of Mathematics*, 2d Series, 18 (1916–17), 200, and need not be repeated here.

II. Weighted algebraic correspondences.

Let A be an irreducible algebraic manifold of one independent complex variable. The projection of A upon the " complex plane " of any complex variable is then a single Riemann surface in the usual sense, while A may be called an irreducible algebraic curve in the domain of birational transformations. Let x denote a variable over A, so that any constant choice assigned to x yields a fixed place on the manifold A. Since the manifold is irreducible, the various possible positions of x can be obtained from a single one by analytic extension. Let B be a second irreducible algebraic manifold with the variable of position, y.

An algebraic correspondence between A and B is a correspondence involving two integers m and n, such that each position x on A determines

n positions y on B, and each position y on B is in turn determined from any one of m positions x on A. A study of correspondences suggests very early the notion of ascribing weight to the various position-pairs in a correspondence. If these all have the same weights the weight might be ascribed to the correspondence as a whole. The weights are frequently positive integers but may have any complex values other than 0. The weight is independent, however, of any continuous variations of the initially determining position so long at least as certain special isolated points be avoided. Thus let x be given at any desired point of A other than the special isolated points explicitly excepted. There are then determined by the correspondence, n points $y_1(x)$, $y_2(x)$, $\cdots$, $y_n(x)$, on B and associated with each pair $(x, y_i(x))$ is a weight c_i independent of variations of x. It may not be possible to assign the weight c_i entirely at pleasure. If by permitting x to make a suitable circuit on A, and avoiding the special points, the x may be brought back to its original place, but two or more y's are interchanged, the interchanged pairs (x, y) will be supposed to have the same weight. The weights are understood indeed to be determined by the x and its associated y's alone, and to be independent of the paths described. It will be assumed that the mere position of x is in all cases sufficient to determine the n positions y. The points x on A that may be associated with a given y on B in the correspondence yield a set of m pairs (x_1, y), (x_2, y), $\cdots$, (x_m, y), each having its own weight. The weight is determined by the pair and the question as to whether the x or the y is regarded as initially assigned is immaterial.

III. Continuous matrices.

A correspondence with weights may be regarded as a continuous real matrix $M(x, y)$. An element or position in the matrix is an arbitrary pair (x, y). The value of an element (x, y) is zero, if the pair are not of positions on A and B respectively, paired under the given correspondence. The value of the element is on the other hand the given weight, if (x, y) form a pair given by the correspondence. A row in the matrix is the totality of pairs (x, y) for which y is fixed, and a column the totality for which x is fixed. Thus a column will contain n, and a row, m non-zero numbers.

Two correspondences between the same two irreducible algebraic manifolds constitute two rectangular matrices $M_1(x, y)$ and $M_2(x, y)$ which may be added or subtracted to yield new matrices or correspondences. A matrix may also be multiplied by a constant. Multiplication by a constant alters only the weights and these proportionately, but does not change the pairs of corresponding elements, providing as usual that the multiplier be not zero.

If a third irreducible algebraic manifold C be taken with z as the corresponding variable of position, a correspondence between B and C will be expressible as a matrix $M(y, z)$. The product $M_1(x, y)$, $M_2(y, z)$ may be formed and a correspondence between A and C will result. In these operations of addition, subtraction, and multiplication the usual rules for finite matrices are observed.

IV. Square matrices.

In particular, correspondences upon a single manifold A will be considered. These may be called square matrices. This will be the case for $M(x, y)$ when B coincides with A. If C be coincident with A then the product $M(x, y)M(y, z)$ will be from x's to z's both on A, while the manifold B may be different. One method of obtaining correspondences upon a single manifold is to start with any $M(x, y)$, where A and B are different, and define $M'(y, x)$ by identity $M'(y, x) \equiv M(x, y)$. If x and y be on the same manifold, that is, if $A \equiv B$, then one may define also $M'(x, y)$, which is then the *transposed* of M. In any case the product $M(x, y)M'(y, x')$ is of the form $M_1(x, x')$ where x and x' are both on A, while $M'(y, x)M(x, y')$ is of the form $M_2(y, y')$ where y and y' are both on B. Both $M_1(x, x')$ and $M_2(y, y')$ are symmetrical, i.e., $M_1 \equiv M_1'$ and $M_2 \equiv M_2'$, and each is obtained from $M(x, y)$ alone. The definition of the product of the square matrices $M_1(x, x')$ and $M_2(x, x')$ is extended so that by M_1M_2 is meant $M_1(x, x'')M_2(x'', x')$.

The most common special matrix is the *identical matrix* for this manifold, of the form $I(x, x')$ where the value of a pair is zero save for $x = x'$, and is then unity.

The notion of a matrix $M(x, y)$ may be extended to some cases in which the correspondence is no longer algebraic, so that in the rows or columns there may be an infinite number of non-vanishing elements. Addition and subtraction present no difficulties but in forming products the additions expressible in the algebraic case in the form

$$\sum_1^p c_{1i}c_{2i}, \qquad i = 1, 2, \cdots, p,$$

are now likely to be of the form of infinite series. The product of two matrices will not be defined unless the series to be summed are (save at most at specified isolated point-pairs with at most an enumerable set of limit pairs) absolutely convergent.

Two square matrices $M_1(x, x')$ and $M_2(x, x')$ are said to be mutually *inverse* if their two products, viz., $M_1(x, x'')M_2(x'', x')$ and $M_2(x, x'')$ $M_1(x'', x')$ are both equal to the identical matrix for this manifold.

It may be noted that so far as the definition is concerned the inverse, M^{-1}, and the transposed M' of a given square matrix M appear to have little in common, M^{-1} being defined by

$$M^{-1}(x, x'')M(x'', x') \equiv M(x, x'')M^{-1}(x'', x') \equiv I(x, x')$$

and M' being defined by $M'(x, x') \equiv M(x', x)$. For algebraic correspondences however these are not unrelated. The algebraic correspondence $M(x, x')$ establishes a relation between x, and x' such that for each x there are corresponding values of x'. What is ordinarily thought of as an inverse is the correspondence determining the values of x that yield a given x', when x' rather than x is given initially. In this sense it is the transposed rather than the inverse matrix that is found. In the terms of functions, if $x' = F(x)$, then $x = F^{-1}(x')$. This implies indeed that in the neighborhood of a point $F(F^{-1}(x)) = F^{-1}(F(x)) = x$. When the total set of values is considered, $F(F^{-1}(x))$ may give other values beside x, as in the case $\log_e (e^x)$ for x complex, giving not only x but also $x + 2n\pi i$ for each integer value of n. Algebraically the resultant of $M(x, x')$ and $M'(x'', x')$ cannot be expected to give merely a power of $(x - x')$ but may contain extraneous factors. An inverse is distinguished from a transposed in that it does not always exist, but if existent accounts for all factors which arise extraneously from the use of the transposed. For algebraic investigation, the study of the inverse naturally presents many interesting features, which cease to be of such importance in the case of infinitely many-valued functions.

For any square matrix M, MM is called the square of M and is written M^2, $MM^2 = M^2M$ is written M^3, etc. If a matrix M has an inverse this is denoted by M^{-1}. A matrix may fail for several reasons to have an inverse, if however an inverse does exist the matrix is said to be *non-singular*.

V. Linear dependence.

A set of k matrices $M_1(x, y)$, $M_2(x, y)$, $\cdots$, $M_k(x, y)$ are defined as being linearly dependent, if there exist k constants, a_1, a_2, $\cdots$, a_k, not all zero such that

$$a_1M_1 + a_2M_2 + \cdots + a_kM_k = 0,$$

where by 0 is meant the matrix all of whose elements have the value zero.

It may happen that a square matrix, $M(x, x')$ singular or not, has the property that the first k powers together with I are linearly dependent so that

$$a_0I + a_1M + a_2M^2 + \cdots + a_kM^k = 0, \qquad a_k \neq 0.$$

The existence of such a relation is not altered if we replace M by any expression of the form $\lambda M + \mu I$ where $\lambda \neq 0$. The coefficients, a, will of course depend however upon the choice of λ and μ. In the above case we have

$$M(a_k M^{k-1} + \cdots + a_1) = (a_k M^{k-1} + \cdots + a_1)M = - a_0 I$$

so that if a_0 be different from zero,

$$- (a_0 M^{k-1} + \cdots + a_1)/a_0$$

is a matrix inverse to M. Thus if such a linear dependence exists the singular matrices of the pencil, $\lambda M + \mu I$ will correspond to at most a finite number of distinct values of the ratio μ/λ, all others being non-singular. To simplify the study of various problems suggested in this connection we shall introduce the notion of " closure matrices " as distinct from the " correspondence matrices " mentioned heretofore.

VI. Closure.

Let there be a correspondence as considered above from x to x' both on the same irreducible algebraic manifold. Let each x determine n values of x'. The following set S of positions all variable with and determined by x_0 will be considered, x_0; $x_1, x_2, x_3, \cdots, x_n$; $x_{11}, x_{12}, \cdots, x_{1n}, x_{21}, x_{22}, \cdots, x_{2n}, \cdots, x_{n1}, x_{n2}, \cdots, \cdots, x_{nn}$; $x_{111}, x_{112}, \cdots, \cdots, x_{nnn}$; $\cdots$, where $x_{rs\cdots tu}$ is one of the values of x' determined when x is taken as $x_{rs\cdots t}$. A more precise determination of the correspondence from $x_{rs\cdots t}$ to $x_{rs\cdots tu}$ need not be examined, so that the order in which the last set of last subscripts 1, 2, $\cdots$, n is assigned in any case need have no special significance. The totality of *distinct* positions of the above infinite set S will be called the *iterative range* R. For a given x_0 when x is any position of R, the corresponding values of x' are also in R. If no two of the members of S are by the nature of the correspondence identical for all positions of x_0, the set S is said to be *totally open*. If there be pairs that coincide but the reduced set in the iterative range R be yet infinite, the set X is said to be *partially open* and *partially closed*. When R is finite, then S is said to be *totally closed*, or merely *closed*.

The discrete $N(x, x')$ obtained from the correspondence so as to have the same weight as in $M(x, x')$ in point-pairs appearing in both matrices, but where the range of variation of x and x' is now the discrete range R, is called the *closure matrix* of the correspondence. The *closure matrix* will be of finite order and thus of the usual type in algebraic discussion if and only if the correspondence is closed.

VII. Valence.

If K be a closed algebraic correspondence on a single algebraic manifold, then the correspondence $K + vI$ will be singular for certain values of v but non-singular for all other values. Should K have the weight one for all pairs save possibly the pairs (x, x') where $x = x'$, the same will be true for $K + vI$. The choice of v may be regarded as so determined that for $v = 0$, K has the weight zero on all save at most a finite number, c, of "coincident" pairs (x, x'), i.e., where $x = x'$.

Now on a given algebraic manifold A correspondences are possible of certain types but not of others. The study of the manifold on which the correspondence is to be constructed is as necessary as the study of the continuous matrix which is to represent the correspondence when constructed. To be sure, the study of the manifold may be regarded as merely the study of the character of the rows and columns of the continuous matrix, since these are simply duplicates of the manifold.

The matrix being regarded as a two-dimensional manifold D, upon which are traced weighted algebraic curves representing the correspondence, the study of possible correspondences K is the study of possible algebraic curves upon the two-dimensional manifold D. The infinite elements of D will be assumed to be adjoined in such a way as to introduce no singularities when bi-rational transformations are effected either on rows or columns. If p be the genus of A, and c be the number of pairs (x, x') where $x = x'$, then for a correspondence between x, and x' of the form considered, where one x determines n x''s and one x' is determined from m x's, it may be shown that when $p = 0$

$$0 = c - (m + n)$$

but for $p \neq 0$, there may be a number v such that

$$2p \cdot v = c - (m + n).$$

In the simplest cases v is a positive integer, but instances may be readily obtained where v is negative or fractional. This value of v is therefore determined by the correspondence K. In the simpler cases the geometric construction of $K + vI$ with this value of v is particularly simple, and for this reason v as so determined is called the *valence* of K.

The notion of valence arises in geometrical discussions, for the first time, in the following manner. Let C_0 be a fixed curve of genus p, cut by a pencil of curves (C) and defining a correspondence as follows: For any simple point x of C_0 there is a unique curve C of (C) meeting C_0 at x in v coincident points, and in n other variable points x', and further in fixed points only. The correspondence thus established is said to be of valence,

v. If a given point, x', is obtained from n possible choices of x, and if c denote, as above, the number of coincidences, the valence v satisfies the relation $2pv = c - (m + n)$. For an extensive discussion of this relation the reader may refer to H. G. Zeuthen, Lehrbuch der Abzahlenden Geometrie, Chapter IV. Some older writers restrict the term valence to the case of positive integers, all others being termed "singular" correspondences.

VIII. Conclusion.

Necessary and sufficient conditions that a weighted algebraic correspondence has a weighted inverse which is also algebraic, the term "inverse" being used in the complete sense here defined, are the following:

1. The correspondence is completely closed. (This insures that an inverse if existent is not infinitely multiple valued but is algebraic.)
2. The correspondence is non-singular in the sense here used. (This insures the existence of an inverse.)

If the correspondence be singular then other correspondences of the type $K + vI$ where K is as given will be non-singular. However, algebraic considerations suggest a definite value of v, the valence, in any given case, for which the correspondence may or may not be singular.

It should be particularly noted that, while the closure matrix has a determinant, the original matrix will not in general have a determinant in any ordinary sense.

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